Dr. Tchebicheff,' said I, what do you think of the quadrature of the circle? Does not such a discovery shake our belief very much in what we call our intuitive convictions about things being possible or impossible?' 'My feeling,' was his answer, 'about the quadrature of the circle is this, that if the Emporer were to give me my choice to be shot at the expiration of a given time if I did not previously find out the quadrature, or else if I did not prove it to be impossible, I would elect to have to discover the quadrature.' [This was in 1874.]

He told me, too, how a pupil of his own, named Lipkin, a freshman at the University of St. Petersburg, had rediscovered Peaucellier's method in ignorance of what Peaucellier had done, and how at his own special recommendation, the Russian government had taken Lipkin in hand, and provided him with means to free himself from all anxiety for a livelihood, and to enable him to pursue his studies at the University with his mind at ease.

The air of Russia seems to no less favorable to mathematical acumen than to a genius for fable and song. Lobacheffsky, the first to mitigate the severity of the Euclidean code and to beat down the bars of a supposed adamantine necessity, was born (a Russian of Russians) in the government of Nijni Novgorod; Tchebicheff, the prince and conquerer of prime numbers, (who, rightly apprehending every fact as an actualized center, a ganglion of intellectual notions, seeks in realistic constructions and problems of every day life the source of his highest and subtlest mathematical inspirations), in the adjacent circumscription of Moscow, and our own Cayley, the central luminary of the mathematical firmament, was cradled amidst the snows of St. Petersburg.

I happened to receive a visit from M. Manuel Garcia immediately after Tchebicheff left me. I showed M. Garcia, who is a zealous geometer, the drawing of the cell and mounting of Peaucellier's instrument left by Tchebicheff. The next day to my surprise and gratification he brought me a working model of it in wood. When I took this instrument under my cloak and showed it after dinner to the Philosophical Club of the Royal Society it drew forth the most lively expressions of admiration from such men as Wheatstone. Tyndall, Sir C. Lyell, Dr. Carpenter, Mr. Busk, Sir B. Brodie, Mr. Playfair, and others. Soon after I exhibited the same model in the hall of the Athenaeum Club to my brilliant friend Sir William Thompson of Glasgow, who nursed it as if it had been his own child, and when a motion was made to relieve him of it replied: 'No! I have not had nearly enough of it—it is the most beautiful thing I have ever seen in my life.'

Had Peaucellier gone boldly to the Institute with his discovery, I feel fully persuaded from my personal knowledge of the men who adorn its lists, that he would not have had to wait nine years or to have depended on my accidental conference with Tchebicheff for his claims to have met with their first suitable recognition."

In a note, Sylvester says Tchebicheff told him he had succeeded in proving the non-existence of a five-bar linkwork capable of producing a perfect parallel motion. Peaucellier's required seven. But again the great Russian was

mistaken ; for fired and inspired by this very lecture, Harry Hart went home and discovered the five-bar perfect parallel motion now known by his name, where for Peaucellier's cell of six bars is substituted Hart's contraparallelogram of four.

ON A METHOD TO CONSTRUCT INTRANSITIVE SUBSTITUTION GROUPS.

By DR. G. A. MILLER.

An intransitive group contains two or more transitive constituents and may be constructed by establishing some isomorphism between these constituent groups. If it contains more than two transitive constituents we may combine them in any way into two constituents and construct the group by establishing some isomorphism between these constituents, where at least one of them is an intransitive group. For example, we can find all the possible groups of degree 11 whose transitive constituents are of degrees 4, 4, 3 by establishing isomorphisms between the intransitive groups of degree 8 which contain two transitive constituents of degree 4 and the transitive groups of degree three. We can also find all these groups by establishing isomorphisms between the intransitive groups of degree 7 whose systems of intransitivity are 4, 3 and the transitive groups of degree 4.

Sometimes we can observe some important group properties from the notation by which an intransive group is represented. For instance, if the group is not simply isomorphic to some one of its transitive constituents it must contain at least two self-conjugate subgroups, differing from identity, that do not have any common operator besides identity. Conversely, if a group contains two such self-conjugate subgroups it may be represented as an intransitive group which is not simply isomorphic to any one of its transitive constituents.

While the method of establishing some isomorphism between two constituent groups seems to be the best general method to construct intransitive groups yet it is sometimes desirable to employ others. To illustrate one of these we shall employ it to find all the possible intransitive groups of degree 10 that contain five systems of intransitivity.

The average number of elements in all the substitutions of such a group is $10-5=5$* and a positive substitution must be of degree 4 or 8, while a negative substitution must be of degree 2, 6, or 10. There is evidently only one such group of order 2. If a group of order 4 contains only positive substitutions it must contain x substitutions of degree 4 and y of degree 8, where $x+y=3$. Hence

*Frobenius, *Crelle*, vol. 101, p. 287; cf. Miller, *Bulletin of the American Mathematical Society*, vol. 2, 1895, p. 75.

$$4x+8(3-x)=4.5\,;\quad x=1,\quad y=2.$$

Since the substitution of degree 4 and one of the substitutions of degree 8 must generate the required group of degree 10 they can have only one transposition in common. Hence there is only one positive group of order 4.

If a group of order 4 contains negative substitutions it must involve one positive substitution besides identity. If this is of degree 4 the two negative substitutions must involve $20-4=16$ elements. Hence one of them must be of degree 6 and the other of degree 10. Since the one of degree 6 and that of degree 4 must generate the entire group there is only one such group. If the positive substitution is of degree 8, the two negative substitutions must involve $20-8=12$ elements. There is clearly one group whose negative substitutions are composed of one transposition and one substitution of degree 10, and one whose negative substitutions contain three transpositions. Hence there are four groups of order 4,—one of these contains only positive substitutions while the other three contain negative substitutions.

If a group of order 8 contains only positive substitutions it must contain x substitutions of degree 4 and y of degree 8, where $x+y=7$. Hence

$$4x+8(7-y)=8.5\,;\quad x=4.$$

The substitutions of degree 4 must therefore generate the entire group. Since at least two of them must have a common transposition such a group must contain a subgroup of order 4 and degree 6. There is only one such group because the remaining generating substitution of degree four must involve the 4 elements that are not found in the given subgroup of order 4.

We proceed to consider the groups of order 8 that include negative substitutions. The positive subgroup of order 4 must be of degree 6, 8, or 10. If it is of degree 6 its substitutions must involve 12 elements. The four negative substitutions of the group must therefore involve $40-12=28$ elements. Since there must be one substitution of degree 10 there can be only one such group. If the given subgroup is of degree 8 the four negative substitutions of the group must include $40-16=24$ elements. There is clearly one group that contains a single transposition and another that contains four substitutions of degree 6.

Finally, if the given positive subgroup is of degree 10 the four negative substitutions must involve $40-20=20$ elements. Hence at least one of these substitutions must be a transposition. If this transposition is found in the substitution of degree 4 the group will contain two transpositions; if it is not contained in this substitution the group will contain only one transposition. We have now considered all the possible cases and found that there are six groups of order 8,—one of these is positive while the other five contain negative substitutions.

The largest possible group of degree 10 that contains five systems of intransitivity is of order $2^5=32$. There is evidently only one group of this order. From this it follows directly that there is only one positive group of order 16 that contains the given systems of intransitivity. The three groups of this order that

contain negative substitutions may be found in exactly the same manner as those of order 8. Hence there are just 16 groups of degree 10 that contain five systems of intransitivity,—1 of order 2, 4 of order 4, 6 of order 8, 4 of order 16, and 1 of order 32. All those that have the same order represent the same abstract group.

Cornell University, November, 1898.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M. (Princeton); Ph. D. (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from May Number.]

PROPOSITION XXXVI. *If any straight XF* (Fig. 44) *makes an acute angle with any ordinate LF* [*of the equisdistantial*], *the point X does not fall without the cavity of the curve, unless previously XF has cut the curve in some point O.*

PROOF. It is certain that the point X may be assumed in XF so near to the point F, that the join LX previously cuts the curve in some point S: otherwise XF either does not fall wholly without the cavity of the curve, and so we have our assertion; or so it does not make with FL an acute angle, rather it would be to suppose that XL comes together with LF in one same straight.

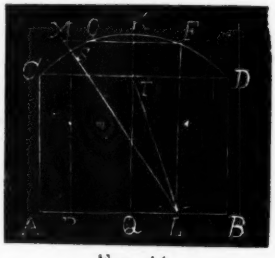


Fig. 44.

Accordingly from the point S let fall to the base AB the perpendicular SP. This will be (from P. 34) equal to LF.

But SP is (from Eu. I, 18) less than LS.

Therefore also LF is less than LS, and therefore much less than LX. Hence in triangle LXF the angle at the point X will be acute, because less (from Eu. I, 18) than the angle LFX supposed acute.

Now let fall to FX the perpendicular LT. This falls (because of Eu. I, 17) toward the parts of each acute angle. Wherefore the point T will lie between the points X and F. Then from the point T let fall to the base AB the perpendicular TQ.

LF (because of the right angle at T) will be greater than LT, and this (because of the right angle at Q) will be greater than QT. Therefore LF will be far greater than QT. But hence, if in QT produced QK is taken equal to LF, the point K (from P. 34) will pertain to the present curve, and therefore the point T falls within the cavity of this curve.

Therefore the straight FT, which cuts two straights QK and LT in T, can

not attain to cutting LS produced in the point X, situated without the cavity of the present curve, unless previously the prolonged FT cuts in some point O the portion of this curve between the points S and K.

Hoc antem erat demonstrandum.

COROLLARY. And hence flows manifestly, that between the tangent of this curve and the curve itself cannot be placed any straight [ray], whether on one or the other side of the tangent, which falls wholly without the cavity of the curve; since a straight [ray] so located must make (from the preceding) an acute angle with the perpendicular let fall from the point of contact to the opposite base.

[To be Continued.]

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

102. Proposed by ALOIS F. KOVARIK, Professor of Mathematics, Decorah Institute, Decorah, Iowa.

A's age is to B's as 2:3. 20 years from now their ages will be to each other as 4:5. What are their ages, respectively?

I. Solution by J. OWEN MAHONEY, B. E., M. Sc., Master of Mathematics and Science, Carthage Graded and High School, Carthage, Texas.

It is easily seen that A's age : A's age + 20 years :: 2 : 4,

or A's age : 20 years :: 2 : 2.

$\therefore$ A's age = 20 years. But B's age = $\frac{3}{2}$ of A's = 30 years.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

In the first instance, A's age = $\frac{2}{3}$ B's age.

In the second instance, A's age = $\frac{4}{5}$ B's age.

$\therefore$ $\frac{2}{3}$ B's age + 20 years = $\frac{4}{5}$(B's age + 20 years).

$\therefore$ $\frac{2}{15}$ B's age = 4 years.

$\therefore$ B's age = 30 years, A's age = 20 years.

An algebraic solution was furnished by Charles C. Cross.

Solutions of problems 98 and 99 were received from J. K. Ellwood, and solutions of problems 100 and 101 were received from J. Scheffer too late for credit in last issue.

ALGEBRA.

89. Proposed by G. A. MILLER, Ph. D., Instructor in Mathematics, Cornell University, Ithaca, New York.

Solve by quadratics, $x^2 + y = 7 \ldots\ (1)$,

$x + y^2 = 11 \ldots\ldots (2)$.

I. Solutions by W. F. BRADBURY, A. M., Head Master Cambridge Latin School, Cambridge, Mass.

The late Professor Quimby, of Dartmouth College, pronounced equations of this form insoluble except by Descarte's formula for bi-quadratics.

Various ways of getting these values of x and y in these particular equations I have seen.

(a) If x and y are integral and positive, from (1) we know x^2 must be 1 or 4; and from (2) we see that x^2 cannot be 1. Hence $x^2=4$, and $y=2$ (not ± 2). Hence $y=3$.

(b) Adding (1) and (2), multiplied by 4, and adding 2 to both members, we have $4x^2+4x+1+4y^2+4y+1=74$, or $(2x+1)^2+(2y+1)^2=74$.

Now if x and y are both integral then 74 is the sum of two perfect squares; the only perfect squares whose sum is 74 are 25 and 49, and as $y>x$ it follows that $(2x+1)^2=25$, and $(2y+1)^2=49$, or $x=2$, and $y=3$. The answers obtained from calling $2x+1=-5$ and $2y+1=-7$ do not prove.

(c) Multiply (1) by x, then $xy^2+x^2=11x$..... (3). From (3) subtract (2), $xy^2-y=11x-7$.....(4). To (4) add twice (1), $xy^2+2y^2-y+2x=11x+15$ .....(5), or $(x+2)y^2-y=9x+15$, or $y=3$, or $(-3x-5)/(x+2)$, and $x=2$, or $x^2=7+(3x+5)/(x+2)$.

This $x^2=7+(3x+5)/(x+2)$ is a cubic equation which gives three other values for x.

II. Solutions by M. E. GRABER, Student in Heidelberg University, Tiffin, O.; W. F. BRADBURY, A. M., Cambridge, Mass.; and P. S. BERG, A. M., Principal of Schools, Larimore, N. D.

Equating values of x^2 from (1) and (2), $7-y=(11-y^2)^2$.....(3), $y^4-22y^2+y+114=0$..... (4).

Factoring $(y-3)(y^3-3y^2-13y+38)=0$.

$\therefore$ $y-3=0$ and $y=3$, and $x=2$. The three other values of y can be found by reducing $y^3+3y^2-13y+38=0$. GRABER, BRADBURY, BERG.

Let $z=y+1$, and substitute for y in the latter disjuntive equation $y^3+3y^2-13y+38=0$, and we get $z^3-16z-23=0$. Assume $z=\omega\sqrt[3]{(r_1)}+\omega^2\sqrt[3]{r_2}$.

$\therefore$ $r_1+r_2=23$, and $\sqrt[3]{(r_1r_2)}=\frac{16}{3}$. $\therefore$ $r_1r_2=\frac{4096}{27}$.

$\therefore$ $r=\frac{1}{2}[23\pm\sqrt{(23^2-\frac{16384}{27})}]=\frac{1}{2}[23\pm\frac{1}{9}i\sqrt{(6303)}]$.

$\therefore$ $y=3$ or $-1+\omega\sqrt[3]{[11\frac{1}{2}+\frac{1}{18}i\sqrt{(6303)}]}+\omega^2\sqrt[3]{[11\frac{1}{2}-\frac{1}{18}i\sqrt{(6303)}]}$ and $x=2$ or $\sqrt{\{11-[-1+\omega\sqrt[3]{(11\frac{1}{2}+\frac{1}{18}i\sqrt{6303})}+\omega^2\sqrt[3]{(11\frac{1}{2}-\frac{1}{18}i\sqrt{6303})}]\}}$.

P. S. BERG.

III. Solution by A. H. BELL, Hillsboro, Ill., and W. F. BRADBURY, A. M. Cambridge, Mass.

Write (1) $x^2-4=3-y$.....(1).

Write (2) $9-y^2=x-2$.....(2).

(1)$\times$(2), etc., $(x^2-4)(9-y^2)-(x-2)(3-y)=0$.....(3).

$\therefore$ We have factors $x-2=0$, $x=2$; $3-y=0$, $y=3$.

The other values of x and y can be easily found. None of these methods can be applied to $x^2+y=a$ and $x+y^2=b$.

IV. Solution by SYLVESTER ROBINS, North Branch Depot, N. J.

Every quantity which is twice a square, as 2, 8, 18, 32, 50, 72, 98, 128,

162, 200, etc., is the sum of two consecutive numbers plus their squares. This is readily seen from $a^2+(a+1)^2+a+(a+1)=2a^2+4a+2=2(a+1)^2$.

Consequently in $x^2+y+x+y^2=7+11=18$, the greater of the two unknowns is $=1/[\frac{1}{2}(18)]=3$, and the less is a unit smaller $=2$.

V. Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

(a) Adding (1) and (2), $x^2+x+y^2+y=18$, or $x^2+x+\frac{1}{4}+y^2+y+\frac{1}{4}=18+\frac{1}{4}+\frac{1}{4}$. $(x+\frac{1}{2})^2+(y+\frac{1}{4})^2=\frac{74}{4}=\frac{25}{4}+\frac{49}{4}$.

Whence $x+\frac{1}{2}=\frac{5}{2}$ or $-\frac{7}{2}$, $y+\frac{1}{2}=\frac{7}{2}$ or $\frac{5}{2}$, or $x=2$, $y=3$, the other answer not being applicable.

(b) From (1), $y-3=4-x^2$; from (2), $y^2-9=2-x$, or $y-3=\dfrac{2-x}{y+3}$.

Whence $\dfrac{2-x}{y+3}=4-x^2$, or $x^2-\dfrac{x}{y+3}=4-\dfrac{2}{y+3}$, or $x^2-\dfrac{x}{y+3}+\dfrac{1}{4(y+3)^2}=4-\dfrac{2}{y+3}+\dfrac{1}{4(y+3)^2}$. $x-\dfrac{1}{2(y+3)}=2-\dfrac{1}{2(x+3)}$,

whence $x=2$, and hence $y=3$.

(c) From (1), $y=7-2$.

Substituting in (2), $x+49-14x^2+x^4=11$, or $x^4-14x^2+x+38=0$.

Factoring $(x-2)(x^3+2x^2-10x-19)=0$,

whence $x=2$ for one root.

The cubic $x^3+2x^2-10x-19$ have 3 real roots, two negative and one positive. The positive root is between 3 and 4; the negative roots between -1 and -2, and -3 and -4.

These are found by Horner's method to be as follows:

$$x_1=3.13+, \quad x_2=-1.74+, \quad x_3=-3.27+.$$

From these the corresponding value of y can be found.

For a different solution see *Fisher and Schwatt's Algebra*, page 576.

VI. Solution by G. H. RICHARDS, Hillsboro, Ill.

(2)$-$(1) $y^2-x^2=4+y-x=1(4+y-x)$, $(y+x)(y-x)=1(4+y-x)$ ....(3).

$\therefore y-x=1$, $y+x=4+y-x$....(4, 5).

(4)$+$(5) gives $2y=5+y-x$. $y=5-x$.

$\therefore x^2-x=2$. $\therefore x=2$, $y=3$.

VII. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let $t=3-y$. This value of $3-y$ in (1) and (2) gives

$$x^2-4=t....(3), \quad x-2=t(3+y)....(4).$$

$\therefore x^2-4=\dfrac{x-2}{3+y}$. $\therefore x^2-\dfrac{x}{3+y}=4-\dfrac{2}{3+y}$.

$$\therefore x^2 - \frac{x}{3+y} + \frac{1}{4(3+y)^2} = 4 - \frac{2}{3+y} + \frac{1}{4(3+y)^2}.$$

$$\therefore x - \frac{1}{2(3+y)} = 2 - \frac{1}{2(3+y)}.$$

$\therefore x=2,\ y=3.$

From (1), $y=7-x^2$. This in (2) gives $x^4-14x^2+x+38=0$.

$\therefore x=2 \qquad y=3$

$x=3.13132 \qquad y=-2.80516$

$x=-3.28316 \qquad y=-3.77914$

$x=-1.84816 \qquad y=3.5843$

$x^4-14x^2+x+38=(x^2-5\frac{3}{9}\frac{3}{9}x+6\frac{2}{9}\frac{6}{9})(x^2+5\frac{3}{9}\frac{3}{9}x+6\frac{6}{9}\frac{6}{8}\frac{4}{0}\frac{}{1})$ approximately.

$\therefore x^2-5\frac{3}{9}\frac{3}{9}x+6\frac{2}{9}\frac{6}{9}=0. \quad x^2+5\frac{3}{9}\frac{3}{9}x+6\frac{6}{9}\frac{6}{8}\frac{4}{0}\frac{}{1}=0.$

$\therefore x=2 \qquad y=3$

$x=3\frac{1}{9}\frac{3}{9}=3.1313 \qquad y=-2.8052$

$x=-3.2832 \qquad y=-3.7792$

$x=-1.8481 \qquad y=3.5843$

Also solved by *W. H. DRANE.*

VIII. Solution by **J. M. BOORMAN**, Consultative Mechanician and Counselor at Law, Woodmere, Long Island, N. Y.

"Empirical" and "Practical" (note page 174, problem 83) each find all the roots by quadratics. "Empirical" (page 173) is here ($fx=x$).

$\therefore \Sigma^4-2(11\pm_1\sqrt{114})\Sigma^2=\frac{1}{4}.(1/_1\sqrt{114}),\ _1\sqrt{114}=\ldots\ldots\ldots 10.677078,252031.$

Complete the square by squaring $(11+_1\sqrt{114})$, which squared

$=125+22_1\sqrt{114}=469.895721,544682$

and adding to both sides

We have $\Sigma^4-2[11+_1\sqrt{(11^2-7)}]\Sigma^2+469.8957$, etc., $=\ldots\ldots 469.919136,189971$

since $\frac{1}{4}.(1/_1\ 114)=\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 0.023414,645289$

Extracting square roots, $\Sigma^2=\ldots\ldots 21.677078,252031+21.677618,323745$

$=43.354696,575776$ and $\Sigma=\pm 6.584428,340849\ldots.(3)$.

True $\Sigma=6.584428,340330,491744,944338,239380,16$, etc$\ldots.(A)$.

Error$=0.000000,000518+$.

To save labor, as (2) transposes to $y^2-9=2-x$, therefore $y-3=[2/(y+3)]-[x(y+3)]$. $\therefore y=3+[2(y+3)]-[x(y+3)]\ldots.(B)$. By (B) in (1) $x^2-[1/(y+3)]x+\square,\ =7-3-[1/(y+3)]2+\square,\ =4-[1(y+3)]2+\square,\ \ldots.(C)$. Extracting the square root and solving for x, $x=\frac{1}{2}\{[1(y+3)]\pm_1\sqrt{[1/(y+3)]}\}\mp(-2)\ldots.(D)$!

$\therefore x=[1/(y+3)]-2\ldots.(E)\ ;\ x_1=2\ldots.(F)$. By (F) in (1) $y_1=3$, $\therefore$ 3 from (A) is $y_a=3.584428,340330,491744,944338,239380,161894,6\pm$

and $x_a=-1.848126,526964,403553,538300,209036,266178,2\mp$

$x_1+x_a=S=(1/\Sigma)=0.151873,47+\ldots.(4)$ and $(4)-2=-1.848126,5+$.

(E) in (1) gives $y^2+[1(y+3)]=13$, whence $y^3+3y^2-13y=38\ldots.(G)$ has the three non-obvious roots. $\therefore x^3+2x^2-10x=19\ldots.(H)$ by parity. But y_a and x_a are known by Σ and S. Therefore easily by quadratics:

$x=3.131312,518250,572965,804300,733409,211483,7\pm$
$y=-2.805118,086952,744853,053572,398087,397376,0\mp$
$x_{11}=-3.283185,991286,169412,266000,514372,745305,5\pm$
$y_{11}=-3,779310,253377,746891,890765,841292,764578,6\mp$

$*S^3\mp_1 2xS^2-2yS=\mp f....(J)$ by $y=\mp_1 Sx+\frac{1}{2}(S^2\pm_1\Sigma)....(K)$ in (I).

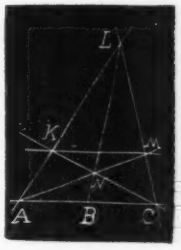
$\therefore x^2\mp_1 Sx=7-\frac{1}{2}(S^2\pm_1\Sigma)$ give by quadratics all x's and y's with no further aid.

GEOMETRY.

102. Proposed by **B. F. FINKEL**, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Ohne Benutzung des Cirkels eine Strecke AC zu halbiren, wenn eine Parallele der Geraden AC gegeben ist. [Reye's Geometrie der Lage, Part I, page 191].

I. Solution by C. HORNUNG, Professor of Mathematics, Heidelberg University, Tiffin, Ohio.



Wir ziehen durch irgend einen Punkt L auserhalb der gegebenen Parallele die Linien AL und CL welche die Parallele in resp. K und M schneiden. Dann ziehen wir die Linien AM und CK die sich im Punkte N schneiden. So bestimmt die Diagonale LN des Vierecks $KLMN$ den gesuchten Halbirungspunkt B der Strecke AC. Denn der vierte von B harmonisch getrennte Punkt, in welchem die zweite Diagonale KM des Vierecks die Gerade ABC schneidet, liegt uendlich fern.

II. Solution by the PROPOSER.

GIVEN the line AC and the line KM parallel to it,

TO BISECT AC without using the circle.

CONSTRUCTION. 1. Through A draw any two lines intersecting the parallel line in the points K and M.

2. Draw the lines CK and CM, CK intersecting AM in N and CM intersecting AK in L.

3. Draw the diagonal, LN, of the 4-side, $LMNK$, and its intersection B with AC will be the required point of bisection.

The points A, B, C, and the point at infinity, *i. e.* the intersection of AC and MK, constitute a harmonic range of points and the point at infinity is harmonically separated from B by the points A and C.

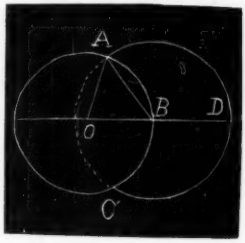
This problem was also solved by *G. B. M. ZERR*, *C. C. CROSS*, *J. SCHEFFER*, *A. F. KOVARIK*, *M. A. GRUBER*, *NELSON L. RORAY*, *ELMER SCHUYLER*, and *JOHN MANIE*.

103. Proposed by **FREMONT CRANE**, Sand Coulee, Mont.

A horse is tethered with a rope which is attached to a stake B on the edge of a circular pond containing one acre. How long must the rope be to allow the horse to graze over one acre? [From *Home Study Magazine*, problem 249].

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; P. S. BERG, B. Sc., Principal of Schools, Larimore, N. D.; WALTER H. DRANE, Graduate Student, Harvard University, and M. A. GRABER, Tiffin, Ohio.

Let B be the point where the horse is tethered, A and C the points on the edge of the given pond, to which the horse can graze, O the center of the pond. Let $AO=r=4\sqrt{(10/\pi)}$, the radius of the pond, and $\angle ABO=\theta$.



$\therefore$ AB=length required$=2r\cos\theta$. Also the area common to the pond and the circle upon which the horse can graze$=r^2(\pi+2\theta\cos2\theta-\sin2\theta)$.

$\therefore$ The area upon which the horse can graze outside the pond is $4\pi r^2\cos^2\theta-r^2(\pi+2\theta\cos2\theta-\sin2\theta)=\pi r^2$. $\therefore$ $2\theta-\tan2\theta=2\pi$.

$\therefore$ $\theta=51^\circ\ 16'\ 24''$. $\therefore$ $AC=2r\cos\theta=8.92926$ rods.

II. Solution by A. H. BELL, Hillsboro, Ill.

Let $10a=1$ acre, be the area of the pond, $10b$ the area grazed over, $R=\sqrt{(10a/\pi)}$ be its radius, and $AB=r=2R\sin\theta$ be the length of rope, where $2\theta=\angle AOB$. Then $r^2=4R^2\sin^2\theta=20a(1-\cos2\theta)/\pi$ .....(1).

2 sector $ABD-2$ segment $AB=10b$, or $r^2(\pi+2\theta)-2R^2(2\theta-\sin2\theta)=20b$. ...(2).

Substituting (1) in (2) and letting $\cos2\theta=x$ and $20b\pi/20a=c$, we get $2\theta x=\pi-c-\pi x+\sqrt{(1-x^2)}$, or

$$2\theta x=\tfrac{1}{2}\pi-c-\pi x+1-\tfrac{1}{2}x^2-x^4/2.4-3x^6/2.4.6-\text{ etc}....(3).$$

By trigonometry,

$$2\theta x=(\tfrac{1}{2}\pi)x-x^2-x^4/1.2.3-3^2x^6/1.2.3.4.5.6-\text{ etc}....(4).$$

(4)$-$(3) and transposing,

$$(\pi-c)+1=y=(3\pi/2)x-\tfrac{1}{2}x^2-x^4/2.3.4-x^6/4^2.5-\text{ etc}....(5).$$

Assume $\cos2\theta=x=ay+by^2+cy^3+dy^4+$ etc....(6).

Substituting (6) in (5) and equating the coefficients of like powers of y, we have $a=2/3\pi$, $b=4/27\pi^3$, $c=16/243\pi^5$, $d=(80+12\pi^2)/(2187\pi^7)$, etc.

Hence, $\cos2\theta=(2y/3\pi)+(4y^2/27\pi^3)+(16y^3/243\pi^5)+[(80+12\pi^2)y^4]/(2187\pi^7\pi)$ +etc. But $c=\pi$, and $y=1$.

$\cos2\theta=.212207+.004778+.000215+.000030+\ldots.=.21723$.

Hence $2\theta=77^\circ\ 27'\ 12'$.

From (1) $r=2R\sin\theta=8.92$ rods.

III. Solution by CHARLES C. CROSS, Libertytown, Md.

Let a=length of rope, b=diameter of pond, and A=area over which the horse can graze. Then $(a^2/6b)(4a+3\pi b)=A$. [See Vol. IV, No. 1, of MONTHLY].

But by the problem $b=2\sqrt{(A/\pi)}$.

Substituting and reducing, we have,

$$a^3+\tfrac{3}{2}\sqrt{(\pi A)}a^2-3\sqrt{(A^3/\pi)}=0.$$

Restoring numbers, $a^3+33.62a^2-3425.514=0$,

whence $a=8.933+$ rods.

104. Proposed by SAMUEL E. HARWOOD, M. A., Professor of Mathematics, Southern Illinois State Normal University, Carbondale, Ill.

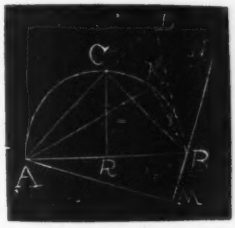
To find a point in a semi-circumference such that the sum of its distances from the extremities of the diameter shall be a maximum. [From *Wentworth's Plane Geometry*, Ex. 387.]

I. Solution by J. F. COWAN, Carbondale, Ill.

Through C, the middle point of the semi-circumference, draw AD, making $CD=CA=CB$. Then BD is perpendicular to AB. Ex. 46, Wentworth.

Through K, any other point in the semi-circumference, draw AN, making $KN=KB$.

Draw NB, producing it to meet AM at right angles.



$\angle ACB=\angle CBD+\angle CDB$.

Exterior angle of triangle BCD.

$\angle AKB=\angle KNB+\angle KBN$.

Exterior angle of triangle KBN.

$\angle ACB=\angle AKB$.

Right triangles.

Triangles CDB and KNB are isosceles.

By construction.

$\therefore \angle CDB=\angle KNB$. $\therefore \triangle ABD$ and $\triangle AMN$ are similar.

$AM<AB$. Base<hypotenuse of triangle.

$\therefore AN<AD$. Homologous lines of similar triangles.

$\therefore AK+KB<AC+CD$. C is the point. Q. E. F.

Mr. Cowan is a student of Southern Illinois State Normal University, and the above demonstration was sent to us by Prof. S. E. Harwood of the Department of Mathematics of that University.

A similar demonstration was given by *P. S. BERG, J. M. COLAW, H. F. STRATTON, WALTER H. DRANE, ALOIS F. KOVARIK*, and *ELMER SCHUYLER*.

II. Solution by J. O. MAHONEY, B.E., M. Sc., Master and Instructor in Mathematics and Science, Carthage Graded and High School, Carthage, Tex.; J. SCHEFFER, A. M., Hagerstown, Md.; CHARLES C. CROSS, Libertytown, Md.; P. S. BERG, B. Sc., Principal of Schools, Larimore, N. D.; G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

From the center R of the semi-circle ACB draw RC perpendicular to AB, then C is the point required.

Proof: ACB is the maximum inscribed triangle.

AKB is any other inscribed triangle.

$AC^2+CB^2=AK^2+KB^2$.

$2AC.CB>2AK.KB$.

$AC^2+CB^2+2AC.CB>AK^2+KB^2+2AK.KB$, or $AC+CB>AK+KB$.

III. Solution by F. W. BRADBURY, A. M., Head Master, Cambridge Latin School, Cambridge, Mass.

Let C be the middle point of the semi-circumference ACB, whose diameter is AB. Draw AC, BC; and CO perpendicular to AB. Then $AC+BC$ is a maximum. $AC>AO$, its projection on AB.

As the point C moves along the arc ACB, CO becomes less, and the line

AC becomes nearer equal to its projection on AB. Just so BC, when C is at A (or C) $AC+BC$=the sum of their projections on AB, or $AO+OB$.

That is, as C moves from the middle of the arc ACB, $AC+BC$ becomes less. $\therefore$ $AB+BC$ is a maximum when C is at the middle of the arc ACB.

IV. Solution by B. F. YANNEY, A. M., Professor of Mathematics, Mount Union College, Alliance, Ohio.

The required point is the mid-point of the semi-circumference.

Proof. Let Q be any other point in the semi-circumference than point P.

Produce AP and BQ until they meet, as at C.

Now $\triangle APD = \triangle BPC$, each being a right triangle, and having a leg and an acute angle of the one=to a leg and a homologous acute acute of the other.

$\therefore$ $PC=PD$.

But $AC>AQ$, and $DB>QB$. Hence $AP+PD+DB>AQ+QB$, or $AP+PB>AQ+QB$. Q. E. D.

Other solutions were furnished by *NELSON L. RORAY*, *CHARLES C. CROSS*, *M. A. GRABER*, *J. M. COLAW*, and *COOPER D. SCHMITT*. Professors Cross, Colaw, and Yanney each furnished two solutions.

CALCULUS.

78. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

Investigate value of $\left(\frac{\tan x}{x}\right)^{1\,:\,x^n}$ where x is 0 and n has consecutive values 1, 2, 3, 4, ... Is there any law governing the different results? When $n=1$, result is 1; when $n=2$, result is $e^{\frac{1}{3}}$; $n=3$, gives ∞, etc.

II. Solution by the PROPOSER.

Let $y=\left(\frac{\tan x}{x}\right)^{1\,:\,x}$, making $n=1$, in the given expression.

Then $\log y=\frac{\log(\tan x\,/\,x)}{x}$ which is of the form $0\,/\,0$.

By the calculus this $=\frac{\frac{x\sec^2 x-\tan x}{x^2}}{\frac{\tan x}{x}}=\frac{x\sec^2 x-\tan x}{x^2}\Big]_{x=0}$ since $\frac{\tan x}{x}\Big]_{x=0}=1$.

The numerator of this expression is the fundamental form which will occur in all the different values given to n.

Continuing the evaluating process we have

$$\frac{x\sec^2 x-\tan x}{x^2}\Big]_{x=0}=\frac{2x\sec^2 x\tan x}{2x}\Big]_{x=0}=\sec^2 x\tan x\Big]_{x=0}=0.$$

Hence $y=e^0=1$.

Let $n=2$, and we have as before,

$$\frac{x\sec^2 x-\tan x}{2x^3}\Big]_{x=0}=\frac{\sec^2 x\tan x}{3x}\Big]_{x=0}=\frac{\sec^4 x+2\sec^2 x\tan^2 x}{3}\Big]_{x=0}=\tfrac{1}{3}.$$

Hence $y=e^{\frac{1}{3}}$.

Let $n=3$, and we have

$$\frac{x\sec^2 x-\tan x}{3x^4}\Big]_{x=0}=\frac{\sec^2 x\tan x}{6x^2}\Big]_{x=0}=\frac{\sec^4 x+2\sec^2 x\tan^2 x}{12x}\Big]_{x=0}=\tfrac{1}{0}=\infty.$$

Then $y=e^{\infty}=\infty$.

Let $n=4, 5, 6\ldots\ldots$ and we have

$$\frac{x\sec^2 x-\tan x}{4x^5},\quad \frac{x\sec^2 x-\tan x}{5x^6},\quad \ldots\ldots\quad \frac{x\sec^2 x-\tan x}{nx^{n+1}}\Big]_{x=0}$$

The numerator will remain as in the former cases, but the denominator will have increasing powers of x. Hence the result will always be ∞.

III. Note by Dr. E. D. ROE, Jr., Norwood, Mass.

In connection with this problem attention might perhaps be called to the fact the indeterminate form 1^{∞} may often be more advantageously dealt with by referring it directly to the foundation of the form, viz, to

$$\operatorname*{Lim}_{\varepsilon=0}\Big(1+\varepsilon\Big)^{\frac{1}{\varepsilon}}=e,$$

where ε is any expression whatever that $=0$, than by the usual method given in the books. Thus in the given example,

$$\frac{\tan x}{x}=1+\frac{x^2}{3}\Big(1+\varepsilon\Big),$$

where ε is a convergent series and therefore $=0$, as $x=0$.

$$\Big(\frac{\tan x}{x}\Big)^{\frac{1}{x^n}}=\left\{\Big[1+\frac{x^2}{3}\Big(1+\varepsilon\Big)\Big]^{\frac{3}{x^2(1+\varepsilon)}}\right\}^{\frac{1+\varepsilon}{3x^{n-2}}},$$

within the limits of convergence and certainly for small values of x. Therefore

$$\underset{x=0}{\text{Lim.}}\left(\frac{\tan x}{x}\right)^{1:x^n}=\underset{x=0}{\text{Lim.}}\left\{\left[1+\frac{x^2}{3}\left(1+\varepsilon\right)\right]^{\frac{3}{x^2(1+\varepsilon)}}\right\}^{\frac{1+\varepsilon}{3x^{n-2}}}=e^{\underset{x=0}{\text{Lim.}}\frac{1+\varepsilon}{3x^{n-2}}}$$

$$=\left(\varepsilon^{\frac{1}{3x^{n-2}}}\right)_{x=0}.$$

This has the advantage of being a direct, closed expression, which shows at once the particular results, 1, $e^{\frac{1}{3}}$, ∞, when n equals 1, 2, $2+r$, where r is any positive integer, and also the general law for any value of n, of whatever character it may be.

MECHANICS.

73. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pa.

A sixteen foot plank weighs thirty-two pounds and is supported by two props, four feet and two feet from the ends. What weight is supported by each prop?

Solution by C. HORNUNG, A. M., Professor of Mathematics, Heidelberg University, Tiffin, Ohio; P. S. BERG, B. Sc., Principal of Schools, Larimore, N. D.; J. SCHEFFER, A. M., Hagerstown, Md.; CHARLES C. CROSS, Libertytown, Md.; ELMER SCHUYLER, High Bridge, N. J.; A. H. BELL, Hillsboro, Ill.; CHARLES E. MEYERS, Canton, O., and M. A. GRUBER, A. M., War Department, Washington, D. C.

Let x=the number of pounds supported by the prop four feet from the end. Then taking moments around the other point of support, we have:

$$(6+4)x=6\times 32, \text{ or } x=19.2 \text{ pounds.}$$

The other prop supports $32-19.2=12.8$ pounds.

Also solved by G. B. M. ZERR.

74. Proposed by B. F. FINKEL, A. M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

In the experiment of swinging in a vertical circle a glass containing water, and suspended by means of a string, if the string be two feet long, what must be the velocity at the lowest point if the experiment is to succeed? [From *Ziwet's Theoretical Mechanics*, Part III., p. 96.]

Solution by the PROPOSER.

Let M=the mass of the glass and water, f=the centrifugal force, r=2 feet =radius. Then $f=Mv^2/r$.

At the highest point, in order that the experiment may be successful $f=Mg$.

$\therefore Mv^2/r=Mg$, whence $v=\sqrt{rg}$.

The velocity, v_1, due to gravity in passing from the highest to the lowest point is $v_1=\sqrt{(2gs)}$, where $s=2r$.

Hence, the velocity, v_2, at the lowest point is,

$$v_2=\sqrt{(v^2+v_1^2)}=\sqrt{(rg+2gs)}=\sqrt{(321.6)}=17.94 \text{ feet per second,}$$

g being equal to 32.16.

Also solved with slightly different results by G. B. M. ZERR, and CHARLES E. MEYERS.

DIOPHANTINE ANALYSIS.

70. Proposed by CHARLES CARROLL CROSS, Libertytown, Md.

Give methods for decomposing numbers into squares, cubes, or biquadates and show that 61×2003 is the sum of ten cube numbers and that 844933 is the sum of eleven biquadrates in thirteen different ways. [From *The Mathematical Magazine*, Vol. II, No. 10.]

Solution by the PROPOSER.

Let s_1=sum of products taking 1 number at a time .....(1),
s_2=sum of products taking 2 numbers at a time.....(2),
s_3=sum of products taking 3 numbers at a time.....(3),
.....
s_n=sum of products taking n numbers at a time.....(n).

Then $a+b+c+d.....n=s_1$ .....(b_1),

$(1)(b_1)-2(2)$ gives

$a^2+b^2+c^2+d^2$ .....$n^2=s_1^2-2s_2$ .....(b_2),

$(1)(b_2)-[(2)(b_1)-3(3)]$ gives

$a^3+b^3+c^3+d^3$..... $n^3=s_1^3-3s_1s_2+3s_3$.....(b_3),

$(1)(b_3)-[(2)(b_2)-(3)(b_1)-4(4)]$ giyes

$a^4+b^4+c^4+d^4$..... $n^4=s_1^4-4s_1^2s_2+4s_1s_3+2s_2^2+4s_4$ .....(b_4),

$(1)(b_4)-[(2)(b_3)-(3)(b_2)-(4)(b_1)-5(5)]$ gives

$a^5+b^5+c^5+d^5$.....$n^5=s_1^5-5s_1^3s_2+5s_1^2s_3+5s_1s_2^2+5s_1s_4-5s_2s_3+5s_5$ ..(b_5),

etc. etc. etc.

[In (b_5) let $s_1=x$, $s_2=-xy$, $s_3=2xy^2$, $s_4=y^2(x^2-y^2)$, and $s_5=\frac{1}{5}y^5$, then $a^5+b^5+c^5+d^5+e^5+f^5=(x+y)^5$. This answers Dr. Drummond's note on page 182, Vol. V., of the MONTHLY.]

71. Proposed by A. H. BELL, Hillsboro, Ill.

Find five numbers such that the product of any two plus 1 will equal a square.

I. Partial Solution by CHARLES C. CROSS, Libertytown, Md.

We readily find four numbers to be $(n-1)$, $(n+1)$, $4n$, and $4n(4n^2-1)$.

Let $n=2$, 3, 4, and so on, and we get

1,	3,	8,	120,	also	1,	8,	15, 528,
2,	4,	12,	420,		2,	12,	24, 2380,
3,	5,	16,	1008,		3,	16,	33, 6440,
4,	6,	20,	1980,		4,	20,	42, 13572,
etc.,	etc.				etc.,	etc.	

From which we get a more general formula for four numbers and find it to be, m, $n^2-1+(m-1)(n-1)^2$, $n(mn+2)$, $m(m^2n-2mn+mn^2+4n-2)^2+(m^2n-2mn+mn^2+4n-2)$.

If there can be found a value for n that will render $(n-1)(128n^3-8n-8)(256n^6-256n^4+63n^2-2n-1)+(256n^6-288n^4-32n^3+65n^2+2n+1)^2=\square$ then the five numbers are $(n-1)$, $(n+1)$, $4n$, $4n(4n^2-1)$, and $[(256n^6-256n^4+63n^2-2n-1)(128n^3-8n-8)]/(256n^6-288n^4-32n^3+65n^2+2n+1)^2$; but life is too short to attempt this.

Dr. Zerr says, "I have not been able to find the five numbers."

If you get no answers to problem 71, I can give Legendre's (improved), and it is a fractional answer for the fifth number. Mr. Wilkes has found four numbers, as follows: $n-1$, $n+1$, $4n$, $16n^3-4n$. Taking these for four of them then I can prove that if there is another integral number it must end in 0.

I have investigated, up to numbers having over five thousand digits, without finding any to answer. A. H. BELL.

I have given considerable time to problem 71, but as yet have failed to obtain five numbers. I have found general values for three such numbers, viz., m, $n(mn\pm2)$, and $(n+1)[(n+1)m\pm2]$. I have also found partial general values for four such numbers, viz., m, $m\pm2$, $4(m\pm1)$, and $16(m\pm1)^3-4(m\pm1)$; but one of the sets of four numbers that I obtained by inspection, can not be obtained by this formula. The set is 1, 8, 15, 528. M. A. GRUBER.

MISCELLANEOUS.

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63. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

Show that the path of a projectile moving with a constant velocity is an inverted catenary of equal strength.

Solution by GEORGE R. DEAN, Professor of Mathematics, University of Missouri School of Mines and Metallurgy, Rolla, Mo.

The differential equations are, obviously,

$$\frac{d^2y}{dt^2}=-g, \quad \frac{ds}{dt}=c.$$

Whence, $\frac{dy}{dt}=-gt+k$, $\frac{dx}{dt}=\sqrt{c^2-(gt-k)^2}$, $x+k'=-\frac{1}{g}\cos^{-1}\frac{gt-k}{c}$.

By division, $\frac{dy}{dx}=\frac{-\cos g(x+k')}{\sqrt{c^2-c^2\cos^2 g(x+k')}}=-\cot g(x+k')$,

or, $\cot^{-1}\frac{dy}{dx}=-g(x+k')$.

Differentiation gives

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{-1}\frac{dy^2}{dx^2}=g.$$

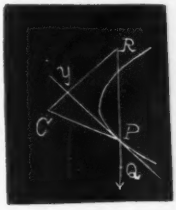
This is the differential equation of the catenary of uniform strength. See Routh's Statics, page 329, or Minchin's Statics, page 315.

64. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Find the caustic by reflection of an hyperbola, the bright point being the center.

Solution by the PROPOSER.

Let RPQ be a reflected ray, Q a point of the caustic where RPQ touches the caustic, R the corresponding point on the secondary caustic. Then RQ is the radius of curvature of the secondary caustic, that is, $RQ=2\rho$, where ρ is the radius of curvature of the locus of y, the foot of the perpendicular from the center C on the tangent at P. Let δ be the perpendicular from C on the tangent at y to the first pedal, then $\delta r=p^2$ (Williamson's Differential Calculus, Art. 188, page 228, Sixth Edition).



Differentiating, $r\frac{d\delta}{dp}+\delta\frac{dr}{dp}=2p$. But $\rho=\frac{pdp}{d\delta}$; and since $p^2(r^2-a^2+b^2)=a^2b^2$ from the hyperbola.

$$\therefore \frac{rdr}{dp}=-\frac{a^2b^2}{p^3}. \quad \therefore \frac{r}{\rho}-\frac{p^2}{r}\cdot\frac{a^2b^2}{p^4r}=2, \text{ or } \frac{a^2b^2}{p^2r^2}=\frac{r}{\rho}-2.$$

Let (m, n) be the coördinates of R, ψ the eccentric angle of P, (x, y) the coördinates of Q. Then

$$\frac{QR}{QP}=\frac{x-m}{x-a\sec\psi}=\frac{y+n}{y-b\tan\psi}=\frac{2\rho}{2\rho-r}.$$

In triangle CPR, $Cy=Ry$, $CP=RP=r$.

$$\therefore \frac{x-m}{x-a\sec\psi}=\frac{y+n}{y-b\tan\psi}=-\frac{2p^2r^2}{a^2b^2}=\frac{2r^2}{a^2-r^2-b^2}.$$

$$\therefore x(a^2-b^2-3r^2)=m(a^2-b^2-r^2)-2r^2a\sec\psi.$$

$$y(a^2-b^2-3r^2)+n(a^2-b^2-r^2)+2r^2b\tan\psi=0.$$

Now $m^2+n^2=4p^2$, $(m-a\sec\psi)^2+(n+b\tan\psi)^2=r^2$.

$$\therefore m=\frac{2ab^2\sec\psi}{a^2\tan^2\psi+b^2\sec^2\psi}, \quad n=\frac{2a^2b\tan\psi}{a^2\tan^2\psi+b^2\sec^2\psi}.$$

But $m(a^2-b^2-r^2)=-m(r^2-a^2+b^2)=-m(a^2\tan^2\psi+b^2\sec^2\psi)=-2ab^2\sec\psi$.

$\therefore\ x(a^2-b^2-3r^2)=-2a\sec\psi(r^2+b^2)=-2a\sec^3\psi(a^2+b^2).$

$y(a^2-b^2-3r^2)=-2b\tan\psi(r^2-a^2)=-2b\tan^3\psi(a^2+b^2).$

By division, $\sin\psi=(y/b)^{\frac{1}{3}}\ /\ (x/a)^{\frac{1}{3}}$.

Eliminating ψ between the last two equations,

$[(x/a)^{\frac{2}{3}}-(y/b)^{\frac{2}{3}}](a^2-b^2-3r^2)^{\frac{2}{3}}=[2(a^2+b^2)]^{\frac{2}{3}}.$

$\therefore\ [(x/a)^{\frac{2}{3}}-(y/b)^{\frac{2}{3}}]^3\,(a^2-b^2-3r^2)=2(a^2+b^2).$

Now $r^2=a^2\sec^2\psi+b^2\tan^2\psi=\dfrac{a^2(x/a)^{\frac{2}{3}}+b^2(y/b)^{\frac{2}{3}}}{(x/a)^{\frac{2}{3}}-(y/b)^{\frac{2}{3}}}$.

$\therefore\ [(x/a)^{\frac{2}{3}}-(y/b)^{\frac{2}{3}}]^{\frac{1}{3}}\{(a^2-b^2)[(x/a)^{\frac{2}{3}}-(y/b)^{\frac{2}{3}}]-3[a^2(x/a)^{\frac{2}{3}}+b^2(y/b)^{\frac{2}{3}}]\},$

$=2(a^2+b^2).$

or $[(x/a)^{\frac{2}{3}}-(y/b)^{\frac{2}{3}}]^{\frac{1}{3}}[(y/b)^{\frac{2}{3}}(\frac{1}{2}a^2-b^2)-(x/a)^{\frac{2}{3}}(a^2+\frac{1}{2}b^2)]=a^2+b^2$, the caustic required.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

104. Proposed by **ALOIS F. KOVARIK**, Instructor in Mathematics and Physics, Decorah Institute, Decorah, Iowa.

If I should buy goods at a price 20% higher than I did buy them, and sell the goods for the same amount that I did sell them, I would gain 25% less than I did gain. What per cent. did I gain? (Solve by Arithmetic).

105. Proposed by **ALOIS F. KOVARIK**, Instructor in Mathematics and Physics, Decorah Institute, Decorah, Iowa.

A teacher looks at his watch when leaving school at noon. When he comes back he finds that the hour hand and the minute hand had just changed places (that they had when he left the school). What time was it when he left, and what time when he came back to school? (Solve by Arithmetic).

*** Solutions of these problems should be sent to B. F. Finkel not later than February 10.

GEOMETRY.

111. Proposed by **GEORGE R. DEAN**, Professor of Mathematics, University of Missouri School of Mines and Metallurgy, Rolla, Mo.

Given that the area of a triangle is equal to half the product of two sides and the sine of the included angle, prove that $\sin(x+y)=\sin x\cos y+\cos x\sin y$.

112. Proposed by **WILLIAM HOOVER**, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio State University, Athens, Ohio.

The tangent planes at A, B, C, D to the sphere circumscribing the tetrahedron $ABCD$ form a tetrahedron $abcd$; prove that Aa, Bb, Cc, Dd will meet in a point if $BC.AD=CA.BD=AB.CD$.

113. Proposed by T. W. PALMER, Professor of Mathematics, University of Alabama, University, Alabama.

Given three concentric circles. Draw a straight line from the inner to the outer circumference that shall be bisected by the middle circumference.

*** Solutions of these problems should be sent to B. F. Finkel not later than February 10.

MECHANICS.

79. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

The four wheels of a street car are rigidly fixed to their axles so that axles and wheels turn together. Is it more advantageous to apply the brakes to the front or to the rear wheels, supposing the brakes to block the wheels in each case?

80. Proposed by B. F. FINKEL, A. M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

A circular board is placed on a smooth horizontal plane and a boy runs with uniform speed around on the board close to its edge. Find the motion of the center of the board.

81. Proposed by JAMES S. STEVENS, Professor of Physics, The University of Maine, Orono, Me.

Two iron spheres whose weights are a and b and a is greater than b, are suspended over a frictionless pulley so that they move in a liquid medium of density δ. Assume that the density of the iron is δ', what would be the spaces passed over (downward by a and upward by b) in the first four seconds, if the spheres start from rest?

*** Solutions of these problems should be sent to B. F. Finkel not later than February 10.

EDITORIALS.

We take this opportunity to express our thanks to our valued contributor, Prof. J. Scheffer, for preparing the index to this volume of the MONTHLY.

Prof. Edgar W. Bass, author of *Elements of Differential Calculus*, and Professor of Mathematics in the United States Military Academy at West Point for the past twenty years, has been placed on the retired list.

This issue concludes the fifth volume of the MONTHLY. The benefit the MONTHLY has done for the progress and advancement of mathematics during the past year must be told by others, but from the many enthusiastic testimonials we have received during the course of the year, the inference is that it is quite material. A number of important articles by some of our foremost mathematicians have appeared during the year. These articles are of the highest interest and value, dealing as they do with the more recent discoveries in mathematics. It is to be hoped that a larger number of contributors in the new fields of mathematical investigation will be added to those who are now carrying on this work; for we believe that no other mathematical journal in this country offers such a great opportunity to reach the large body of teachers of mathematics as does the MONTHLY.

For example, *The American Journal of Mathematics* has a very limited constituency in this country, the reason being that it deals with only the highest subjects of mathematics and the treatment of these subjects are too abstruse and recondite to be read by any but the most advanced students of mathematics. *The Annals of Mathematics* being of a slightly less advanced character appeals to a larger body of readers, and therefore exerts a wider *influence* on the development of mathematics among us.

Five years ago, it seemed that the time had arrived when another mathematical journal could be started, which would draw to its support all persons interested in the unraveling of intricate problems in the various branches of mathematical as well as those who had gone far beyond this stage into the field of original research. It was believed then, and five years experience in the work has confirmed the belief, that a journal devoting about half its space to the solution of problems, and the other half to mathematical papers by acknowledged authorities in the various lines of original investigation, would be of great value. In a few cases we have published some papers from considerations other than scientific. We admit that such considerations are unfortunate, but in the editing of a scientific magazine, there are not unfrequently problems proposed more difficult of solution than merely deciding as to the merits of any particular article for publication. The editor of the MONTHLY has had many of these to solve. However, the appearance of unscientific articles is very rapidly diminishing, and we hope that none will appear in the future.

An apology for a department of Problems and Solutions in the MONTHLY may be found in Professor Elliott's address before the London Mathematical Society on November 10. Among other things, he says, "Secondary work is necessary that the transition from narrow to widened views of Mathematical opportunity be effected surely and without discouragement. The passion among us for examination into elegant incidentals, which shows itself in the fascination exercised by problem making and solving, must be reckoned with, and in my opinion not discouraged. Unambitious work of definitely educational intention in subjects now made known to the select few by ambitious treatises is needed. Unassuming partial and introductory books of didactic character on modern subjects are wanted." We make mention of this, because our best mathematicians have treated too lightly the importance of well chosen problems as means to stimulate the inquirer to press his investigations into the vaster regions beyond the stage of mere problem solving. If those who are sitting in the dazzling light of truth would only condescend to hand down some of the truths so very clear to them in a form easily translated by those sitting in darkness, vast improvement in mathematical teaching would quickly follow. Only quite recently has there been a move in this direction. Men of acknowledged ability are now beginning to see that the way to improve the teaching of mathematics is to improve the works in the elementary branches. To this work, Professor Klein is now devoting much of his time, and it is to be hoped that many of the great teachers and creators of modern mathematics may follow his example.

There seems to be room for one more mathematical journal in this country. It is a journal particularly adapted to the wants of that large body of teachers of mathematics in our High Schools. What sort of a journal this body of workers want we are unable to say, but true it is that very few of them read any of the mathematical journals now published in this country. Not more than half a dozen are found on our list of subscribers while we believe that not so many are found on the lists of the *American Journal of Mathematics* and *The Annals of Mathematics* together. There is not one reason but a number of reasons why this condition exists, a discussion of which would lead us entirely beyond the limits of a mere note. But one reason is that the great majority of the positions in our high schools are filled without any special reference to the qualification of the applicant to teach mathematics, it being a prevalent opinion that any one who has the multiplication table well at hand is quite competent to teach mathematics, and thus many of our high school positions are filled by persons who are as absolutely incompetent to teach mathematics as is a person who only knows the Greek alphabet to teach Greek. We are not saying that teachers of mathematics in our high schools do not read any literature on the subject they teach. Many of them read quite a good deal, but if some of this reading were devoted to some good, live mathematical journal instead of reading the extended discussions of the "idiotic" Grube Method and kindred methods found in the ordinary educational journals, it would be better for the teacher and infinitely better for his pupils. Hence, if these positions were filled by persons who are in love with their subject, a journal of the scope of the MONTHLY might meet their wants.

If every college and university would follow the principle established by one of our largest universities, viz., to fill mathematical positions with first-class teachers of mathematics, a very necessary improvement would follow in a very short time. Not infrequently are mathematical positions filled by persons who, while in college, took an extended course in latin, or Greek, or History, and their work in mathematics not extending beyond trigonometry. This is always done to the great detriment of mathematics both as a science and as an art.

While such conditions exist, it is desirable that every opportunity available should be siezed to make improvements in the teaching of mathematics, by, as much as possible, filling positions in mathematics with good teochers of mathematics, and by improving the methods of presenting the fundamental principles of the science in our elementary text-books.

The MONTHLY offers at present the best opportunity to disseminate sound doctrine in the elementary branches, and if some of our best mathematicians would take up this work, great and lasting good would result.

We take this opportunity to thank Dr. Miller for the good work he is doing in discussing the Group Theory; to Dr. Halsted for his continued articles on Non-Euclidean Geometry; to Dr. Roe for his articles on Symmetric Functions; and to all others who have contributed articles, problems, and solutions. We shall rely on your continued support during the coming year, and by united effort much may be accomplished to create a larger interest in mathematics.

Efforts are being made to make the MONTHLY a permanent factor in the development and progress of mathematics in the United States, either by endowing it so that it may remain where it is now or by transferring it to some university where ample provision will be made for its perpetual maintenance.

BOOKS AND PERIODICALS.

A Treatise on Roofs and Bridges with Numerous Exercises. By Edward A. Bowser, LL. D., Professor of Mathematics and Engineering in Rutgers College. 8vo. Cloth. 195 pages. New York: D. Van Nostrand Co.

This little work is in line with the other works written by Dr. Bowser. Logical order, simplicity of treatment even with a difficult subject are the prominent features of the work. The principles and methods employed in finding the forces in Roofs and Bridges are clearly explained. The whole subject is discussed and developed in accordance with the best methods used in the modern practice of Roof and Bridge construction.

B. F. F.

The American Monthly Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year in advance. Single numbers, 25 cents. The American Monthly Review of Reviews Co., 13 Astor Place, New York.

The question of the bearings of our federal Constitution on the government of newly acquired territories, about which so much haze seems to have gathered in the popular mind, is very clearly and exhaustively treated in the *American Monthly Review of Reviews* for January by Professor Harry Pratt Judson, of the University of Chicago. Professor Judson reaches the conclusion that the Constitution presents no difficulties whatever to our acquisition and control of such territories as the Philipines, for example.

B. F. F.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1.00 per year in advance. Single number, 10 cents. Irvington-on-the-Hudson.

Julian Ralph, who has been for some years in England, writes for the December *Cosmopolitan* a very clever analysis of what seems to him the English ideas of a gentleman; and Mr. John Brisben Walker attempts to consider the American ideals. We are in the formative stage of American manners, and too much stress can scarcely be given to the dangers of introducing those ideas which are least admirable in th character of our English cousins.

B. F. F.

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Other magazines not mentioned above may be obtained from us at reduced rates.

INDEX.

ALGEBRA (See Solutions and Problems).
ARITHMETIC (See Solutions and Problems).
BIOGRAPHIES—
Réné Descartes, with portrait, by B. F. Finkel 191–195
Bolyai Janos (John Bolyai) by George Bruce Halsted 35–38
Tchebychev, by George Bruce Halsted 285–288
BOOKS, Reviewed, 32-34, 64-66, 97-98, 120-124, 159-160, 188-190, 219-220, 247-249, 283-284, 308.
School Algebra Complete, by Fletcher Durell and Edward Robins, 32; *The Equations of Hydrodynamics*, by Joseph Collier, 33; *A New Astronomy*, by D. P. Todd, 33; *An Algebraic Arithmetic*, by S. E. Coleman, 33; *College Algebra*, by Edward A. Bowser, 64; *Elements of Calculus*, by James M. Taylor, 64; *Elements of Determinants*, by Paul H. Hanus, 64-65; *Exercises in Choice and Chance*, by William Allen Whitworth, 65; *Through Quadratic Equations*, by Jos. V. Collins, 65; *Theoretical and Practical Graphics*, by Frederick N. Willson, 120; *New Psychology*, by John P. Gordy, 122; *Mechanical Drawing*, by J. C. Tracy, 122; *Elementi di Geometria*, by G. Lazzeri e A Bassani, 123; *Lessons in Elementary Mechanics*, by Sir Philip Magnus, 160; *Elements of Trigonometry with Tables*, by Herbert C. Whitaker, 188; *Prismoidal Formulae and Earthwork*, by Thomas U. Taylor, 188; *Lecture on the Geometry of Position*, by Theodore Reye, 188; *Elements of Plane and Spherical Trigonometry*, by J. W. Nicholson, 219; *Introduction to Algebra*, by G. Chrystal, 219; *Infinitesimal Analysis*, by William Benjamin Smith, 220; *Advanced Mechanics*, Vol. I Dynamics, by William Briggs, 247; *Coordinate Geometry*, by Briggs and Bryan, 247; *Mensuration and Spherical Geometry*, by William Briggs, 248; *Differential and Integral Calculus*, by James M. Taylor, 248; *Differential and Integral Calculus*, by P. A. Lambert, 248; *Text Book of Algebra*, by George Egbert Fisher, 248; *Introduction to the Theory of Analytic Functions*, by J. Harkness and F. Morley, 249; *An Elementary Course in Analytical Geometry*, by J. H. Tanner and Joseph Allen, 249; *Lectures on Elementary Mathematics*, by Lagrange, 283; *A Manual of Experiments in Physics*, by Jos. S. Ames and William J. A. Bliss, 283; *Elements of the Differential Calculus*, by James McMahon, 283; a Treatise on Roofs and Bridges, with numerous exercises, by Edward A. Bowser, 308.
DIOPHANTINE ANALYSIS (See Solutions and Problems).
EDITORIALS, 30-32, 64, 159-160, 187-188, 219, 282-283, 305-308.
ERRATA, 34, 66, 98, 126, 220, 250.
GEOMETRY (See Solutions and Problems).
MATHEMATICAL PAPERS—
Burnham, A. C., On some Cases when the Quintic is Solvable by Elementary Methods 99–101
Calderhead. James A., New and Old Proofs of the Pythagorean Theorem 73–74
Dickson, L. E., A New Solution of the Cubic Equation 38–39
Halsted, George Bruce, Non-Euclidean Geometry: Historical and Expository 1-2, 67-68, 127-128, 290–291
The First Award of the Lobachevski Prize 39–41
Translation of Monsanto's Teorema 265–266
Heaton, Henry, Infinity, the Infinitesimal, and Zero 224–226
Landis, W. W., A Method of Defining the Ellipse, Hyperbola, and Parabola as Conic Sections 72–73
Lovett, Edgar Odell, Sophus Lie's Transformation Groups 2-9, 75–82
Miller, Dr. G. A., Note on the Application of a Substitution Group in Spherical Trigonometry 102–103

On Several Points in the Theory of Groups of a Finite Order 196–199
On the Groups which are Determined by a Given Group 221–224
Monsanto, J. M., Teorema, Translation by Dr. Halsted 265–268
Moulton, F. R., On the Best Method of Solving the Markings of Judges of Contests 68–72
Nicholson, J. W., Reply to Professor Fisk's Criticism of a Certain Feature of Nicholson's Calculus 199–201
Roe, E. D., Jr., Note on a Formula of Symmetric Functions 161–164
Rothrock, David A., Point Invariants for the Finite Continuous Groups of the Plane 249-264
Yanney, B. F., New and Old Proofs of the Pythagorean Theorem 73–74
Zerr, G. B. M., Summation of Series 128–135
Convex Surface and Volume of Conical Ungulae 164–170

MISCELLANEOUS (See Solutions and Problem).

NOTES, 62-64, 125-126.

Note on the synchronous setting or rising of the cusps of a crescent moon, by S. Hart Wright, 27; The first award of the Lobachevski Prize, by George Bruce Halsted, 39-41; Note on second solution of problem 49, Mechanics, by Alfred Hume 51; A note from McLellan and Dewey's Psychology of Number and from Lefevre's Number and Its Algebra, 57; A note on Euler's attempts of finding triangles of commensurable sides and medians, 59; Notes on the Evanston meeting of the American Mathematical Society, by G. A. Miller, 62-64; Note on the practical application of a substitution group in spherical trigonometry, by G. A. Miller, 102-103; Note on an extract from Dr. Laisant's book, La Mathematique, by Dr. Alexander Macfarlane, 125-126; Note on Mr. J. F. Travis's solution of problem 87, Arithmetic, 139; Note on J. M. Boorman's solution of problem 78, Algebra, by Josiah H. Drummond, 174; Note on solution of problem 69, Calculus, by Josiah H. Drummond, 177: Note on Professor Zerr's solution of problem 60, Diophantine Analysis, by Josiah H. Drummond, 182; Note by the editor, B. F. Finkel, as answer to a query, 202; Note on a demonstration of problem 98, Geometry, by B. F. Finkel, 234; Reply to note of Dr. Drummond on problem 78, Algebra, by J. M. Boorman, 270.

PORTRAITS, Leonhard Euler, facing page 1; Réné Descartes, facing page 191.

PERIODICALS, 34, 66, 97-98, 124-125, 160, 190, 220, 308.

SOLUTIONS AND PROBLEMS.

ARITHMETIC.

A and B traveling at uniform and stated rate, B turning twice, to find B's rate. No. 87 103
A barn 20 feet square, standing in a pasture, horse tied by a rope of 50 feet tied to a corner, find area of land grazed over. No. 93 170–171
A chaise turning within a ring, to find the circumference. No. 85 11
A man gained 3% on his money in July, lost 2% in August, what per cent. of his money July 1, is his money September 1. No. 101 267–268
An interest-bearing note of $1000 with several indorsements, find amount due at a stated time. No. 91 138
A owes $6000, bearing 6% interest, debt to be paid in 6 equal payments, with certain other conditions, to find amount of each payment. No. 90 136–138
A's age is to B's age as 2:3, etc. No. 102 291
Debt of $20 paid in 11 monthly installments of $2 each, to find rate per cent. No. 98 ..... 202, 227
Find greatest number of inch balls placed in a box of given dimensions. No. 89 136
Find principal of an interest-bearing note, payments being made at different stated times, the final amount and the date when due being given. No. 88 104

Find 24 triangles containing 330 square yards. No. 84.....10, 82–85
Income realized by U. S. 4% bonds at 105, sold after 6 years at 104. No. 94 171–173
In what time will $4000 amount to $1534.96, 6% interest, payable annually. No. 97 202
Number of acres in a square field, whose diagonal is 10 rods longer than the side. No. 96 201–202
Stock bought at 4% discount, sold at 2½% premium, brokerage in both cases, ¼%, net profit being $130, find investment. No. 100 267
300 cats kill 300 rats in 300 minutes, how many cats will kill 100 rats in 100 minutes. No. 99 227

ALGEBRA.

A Savings and Loan Company loaned out a certain sum under some particular terms. The question is whether the Company sustains any loss. No. 88 268–270
A starts around a circular island at a certain rate, B, C, and D start at different times at given rates in opposite directions, to find size of island, when they meet, etc. No. 87..... 229–230
If, on the present electoral basis, all electoral votes of each state are cast solid for one or the other candidates, find all possible combinations for 273 votes for winning candidate. No. 84..... 203
Of n persons, A gives to the others as much as each one has, so B, C, etc. What had each at first. No. 79 85–86

Prove identities $2-\sqrt{2}=\frac{1}{2}+\frac{1}{2^2.3}+\frac{1}{2^3.3.17}\ldots\ldots$ and $\frac{5-\sqrt{5}}{2}=\frac{1}{1}+\frac{1}{3}+\frac{1}{3.7}+\ldots\ldots$ No. 76 14–17

Show $\frac{a_1{}^r}{(a_1-a_2)(a_1-a_3)\ldots(a_1-a_n)}+\frac{a_2{}^r}{(a_2-a_1)(a_2-a_3)\ldots(a_2-a_n)}+\ldots$ is 0, if $r<n-1$, $=1$, if $r=n-1$, and to Σa_n, if $r=n$. No. 81 ..... 105-106, 129–141
Solution of two simultaneous equations by quadratics. No. 89 301
Solve equation $(6x^2+x-3)^2-48^2=(x+15)^2$. No. 77..... 17
Sum $\frac{1}{1^2.2^2.3^2.4^2.5^2}+\frac{2}{2^2.3^2.4^2.5^2.6^2}+\ldots$ No. 85..... 203–205
$x^2+yz=16$, $y^2+xz=17$, $z^2+xy=22$. No. 86..... 227–229
$x^2+xy=10$, $y^2+xy=15$. No. 78..... 42–45
$1+x^4=a(1+x^4)$. No. 80 87
$y^2+yz+z^2=a^2$, $z^2+zx+x^2=b^2$, $x^2+xy+y^2=c^2$. No. 82..... 106–107
$x^2+y=8$, $y^2+x=60$. No. 83..... 173–174

GEOMETRY.

Area grazed over by a horse tethered to a post on the circumference of a circle. No. 103 296
Bisectors, if two are equal, the triangle is isosceles. No. 85..... 108–109
Bisectors of angles of opposite sides of inscribed quadrilateral cut sides at vertices of rhombus. No. 90 143
To bisect a line without use of circle. No. 102 295
Circle touching another internally, and a third with a radius, a mean proportional between radii, through point of contact, intersection parallel. No. 80..... 18
Circle tangent to three circles. No. 89 142–143
Circle to be constructed, passing through two given points and touching a third. No. 98..... 234

Circle, to find point in semi-circumference, sum of distances to be a maximum. No. 104 297
Collinear, center of circles being. No. 81 45–46
Escribed circles, radius of circle through centers of inscribed and two escribed circles equal to radius of circumscribed circle. No. 97 231–234
Escribed circles, centers given, and centers of inscribed circles. Prove, etc. No. 100 271–273
Extremities of base of triangle joined to extremities of angle of squares erected on two sides. No. 82 19–20
Focus, four conics having common. No. 86 109
Frustum of cone, find volume of. No. 88 109–110
Isosceles triangles described on three sides of triangle. No. 96 206–207
Locus of point, when coördinates are, as given. No. 83 46
Locus of vertices of triangle. No. 91 174–175
Locus of point, trisecting arcs having common chords. No. 84 87–89
Locus of vertices of all right cones having same ellipse as base. No. 99 270–271
Mountain, finding height of, angles of elevation and base being given No.93 175–176
Parabola, at each point of which the rectangular hyperbola described of four pointic contact, locus of center equal parabola. No. 95 206
Perimeter, to be a minimum. No. 87 141–142
Quadrilateral, inscribed in circle. Prove $AB.BC+AD.CD:AB.AD+BC.CD=BD:AC$. No. 92 175
Tower, finding height of. No. 93 175–176

CALCULUS.

Augur hole, bored through cylinder, find volume removed. No. 63 20–23
Differential equation, given, to prove certain relations. No. 68 110
Differential equation of third order, to solve. No. 71 ..... 145-146, 176
Differential equation, given, to solve. No. 75 235–236
Differential equation, given, to solve. No. 76 236–237
Drawbridge, across which, when moving, a man walks. No. 67 89–91
Elliptic fence, to which a horse is tethered. No. 69 111
Integral, prove given, with conditions stated. No. 70 143–145
Integral, given, to solve. No. 73 207–208
Illusory value when $x=0$. No. 78 298
Park, forming parabolic segment. No. 72 177
Pedal of bicycle, find equation of a point of. No. 77 237
Rope, wound around a conical frustum, how far will a hawk fly unwinding it. No. 66 47–48
String, wound spirally around a cone, distance a duck must swim through, unwinding it. No. 65 46–47

MECHANICS.

Ball, falling through center of earth, its motion similar to that of a pendulum. No. 72 276–277
Beam is suspended by ropes from a horizontal support, angles to be found made by the ropes and support. No. 63 149–150
Body, sliding from rest down a series of smooth inclined planes, final velocity to be found. No. 54 24
Conical stick of timber, being depressed in a liquid of given density, amount of ascent to be found. No. 67 238–239
Cow, jumping over the moon. No. 56 91
Cylindrical vessel, filled with water, resting on horizontal plane, find maximum angle of elevation to which the plane must be raised. No. 64 209–213
Distance from mid-point of a chord of parabola being constant. Locus of center of gravity of segments. No. 66 238

Disturbing force of Moon for any point on earth's surface. No. 68 239–240
Elastic string, by which a body is suspended from a fixed point. Find depression of body, so that when let go, the point of suspension is reached. No. 61 146–148
Endless uniform chain, hanging over two small pegs, to show that in a state of equilibrium a certain given relation prevails between distances of the vertices of the two catenaries, the length of the chain and angle of inclination. No. 58 92–93
Gold, a sphere of, whose density varies as square of distance from center, how to be divided into three parts of equal value. No. 71 275–276
Homogenuous sphere, having given angular velocity and contracting by cooling, find angular velocity when radius is reduced to one-half. No. 75 275
Intersection of two planes, over which a chord slides, find equasion of path described by center of gravity. No. 57 111–112
Isosceles triangle of vertical plane and horizontal base, supported at each end, representing 3 rods joined together. A load is suspended at one vertex, find angles between sides and base on condition, that sum of weights of rods is a minimum. No. 64 177–179
Locus of center of gravity of all segments formed by chord in parabola. No 65 179
Particle of mass moving, in circumference of ellipse in the constant rate, where it is held by attractive forces in the foci. No. 62 148–149
Radius of a sphere of given specific gravity, which will rest in fluid varying density. No. 59 112–113
Ratio of the two legs of a uniform and heavy right triangle, being suspended from center of inscribed circle. No. 60 113–114
Spheres, three equally heavy, are placed on a rough ground and a fourth one on top of them. No. 55 ..... 24, 48–50
Velocity, to find so that water may spill in swinging a pail with a rope. No.74 300
Weight, supported by two props. No. 73 300

DIOPHANTINE ANALYSIS.

Arrangements, greatest number of nine digits, all taken together, whose 3 terminal figures shall be those of a square number under conditions set forth in the problems. No. 57 93
Base of right triangle being 105, find all perpendicular and hypotenusus to fit it, such that their values shall be integers. No. 58 51–54
Consecutive numbers, four, whose sum=square, and sum of whose squares= square. No. 61 215–216
Cubic proper fractions, two, whose product=square proper fraction. No. 66. 215
Decomposing number, give methods of, into squares, cubes, biquadratic. No. 70 279–301
Five numbers to be found, so that the product of any two +1=square 301
Four square numbers in arithmetical progression. No. 62 180–181
General value for p in $4p+1$=sum of 2 squares. No. 68 240–243
Infinite series of prime, integral, rational, scalene triangles where sides of every term are consecutive numbers. No. 61 150–152
Integars, number of equal, and prime to it, show, etc. No. 56 26
Magic squares, whose sum=$3m$. No. 55 25–26
Required to take from the proper key suitable material and construct a "nest" of 10 or 15 prime, integral, rational trapeziums. No. 74 181–182
Six positive numbers to be found, such that if each be diminished by five-half times the fifth power of this sum, the six remainders will be rational fifth powers. No. 60 114
Sum of mth power of all numbers less than P and prime. No. 59 93–94

Show, that 1521 can be expressed in seven different ways as sum of three perfect squares. No. 75 214–215
Two right triangles have same base which is mean proportional between the two perpendiculars, find general solution giving integral values for all the sides. No. 69 277–279
$x^3 + y^3 = 20^3 \times 105489$, find four positive integral values. No. 63 181

AVERAGE AND PROBABILITY.

Average area cut from a circle, from a point on the surface of which 2 lines are drawn to the circumference. No. 58 114–115
Average area of a triangle, in which is a given point and a random point, the line through both divides triangle into a trapezium and a triangle. No. 64 213–214
Average velocity of a point in the circumference of a circle, rolling along a line. No. 59 115–116
Bag, containing any number of balls, which may be either black or white, one is drawn which is white, show that chance of drawing another white one is two-thirds. No. 62 213
Chance that center of gravity of a triangle lies inside a triangle formed by 3 points taken at random within the triangle. No. 56 54–55
Chance that, when a chord is drawn through 2 points in a circle, and a second chord through 2 other points, the quadrilateral formed by extremeties of the two chords contains center of circle. No. 57 56–57
Four points are taken at random within an ellipse, chance that they form a re-entrant quadrilateral. No. 60 152–154

MISCELLANEOUS.

Caustic by reflexion. No. 59 ..... 156-157, 182-183, 303
Chronometers, of three, one keeps true time, two others gain and lose, find true time, when hands of other two point out certain numbers. No. 63 244–245
Conical bin of given dimensions is cut by a plane parallel to side and passing through center of base, how many bushels of wheat will it contain. No. 64 279–280
Cusps of moon's crescent: how long will they set synchronously, the latitude and declination of sun and moon being given. No. 54 and No. 56 ..... 27-28, 116
Multiply 6×4, is the problem legitimate, when both sysbols represent pure numbers. No. 55 57–58
North Latitude, the highest, in which the sun will shine in at the north window of a building at least once a year. No. 53 26–28
Particle is placed very near the center of a circle, round the circumference of which n equal repulsive forces are symmetrically arranged, each force varying inversely as the mth power of its distances from particle, show resulting force, etc. No. 57 116–118
Path of projectile, moving with constant velocity is an inverted catenary of equal strength. No. 65 302
Product, the, of n numbers, each the sum of 4 squares, may be expressed as sum of 4 squares in $(48)^{n-1}$ different ways. No. 61 184
Shadow, the longest in winter at noon, of an upright is found to be seven times as long as the shortest summer shadow of same object. No. 58 155–156
Sun's declination being given, in what latitude will it shine on north side of building during first half of forenoon, and on south side during the other half. No. 60 183–186
Triangles, whose sides and median lines are commensurable. No. 42 58–59
Tube of uniform cross section is bent into the form of a cycloid, equal quantities of fluids of different specific gravity are poured in at the two cusps, find distance of upper levels of fluids from vertex. No. 62 243–244

INDEX.

ALGEBRA (See Solutions and Problems).

ARITHMETIC (See Solutions and Problems).

BIOGRAPHIES—

Sophus Lie, with portrait, by George Bruce Halsted 97–101

Dr. Percival Frost, with portrait, by George Bruce Halsted 189–191

BOOKS, Reviewed, 23–24, 51–52, 95–96, 128, 159–160, 187–188, 217–218, 253–254, 291–292, 312

A Text-Book of Statics, by William Briggs, 23; *Elements of Trigonometry*, by Andrew W. Phillips and Wendell M. Strong, 23; *On the Study and Difficulties of Mathematics*, by Augustus De Morgan, 23; *Plane and Solid Geometry*, by James Howard Gore, 51; *Problemes de Geometrie Elementaire*, by Ivan Alexandroff, 52; *Sur la Resolution de l'Equation du Troisieme Degre*, by Alexander Macfarlane, 52; *Calcul de Generalization*, by G. Oltramar Doyen, 52; *Tables of Logarithms to Five Places*, by Edwin S. Crawley, 95; *A Short History of Astronomy*, by Arthur Berry, 95; *Mathematical Essays and Recreations*, by Herman Schubert, 95; *The Elements of Physics*, by Alfred Payson Gage, 96; *A Text-Book of General Astronomy for Colleges and Scientific Schools*, by Charles A. Young, 96; *A Text-Book of Physics*, by G. A. Wentworth, and G. A. Hill, 96: *An Introduction to the Differential and Integral Calculus and Differential Equations*, by F. Granville Taylor, 128; *A Text-Book of General Physics*, by Charles S. Hastings, 128; *A Text-Book of Physics*, by J. H. Pointing, 159; *The Elements of Graphic Statics*, by L. M. Hoskins, 159; *Rentgen Rays*, Memoirs of Rontgen, Stokes, and Thompson, 159; *A History of Physics*, by Florian Cajori, 160; *The Modern Theory of Solution*, Memoirs by Pfeffer, Van Hoff, Arrhenius, and Raoult, 160; *Algebra Elementare*, by Nassolti, 160; *Bibliography of American Arithmetic*, by J. M. Greenwood, 187; *A Primer of the Calculus*, by E. Sherman Gould, 187; *School Algebra*, by George Egbert Fisher, and Isaac J. Schwatt, 188; *A Treatise on the Theory and Solutions of Algebraical Equations*, by John Macnie, 217; *New Higher Algebra*, by Webster Wells, 217; *An Elementary Treatise on the Theory of Equations*, by Samuel Mars Barton, 217; *Memoir on Substitution Groups*, by G. A. Miller, 217; *A Report on Greene County*, by Edward M. Shepard, 217; *The Fundamental Principles of Algebra*, by Alexander Macfarlane, 218; *Colorado College Studies*, 253; *Plane and Solid Geometry*, by William J. Milne, 253; *Discourse on Method*, by Rene Descartes, 253; *New Plane and Solid Geometry*, by Wooster Woodruff Beman, and David Eugene Smith, 253; *Elementary Illustrations of the Differential and Integral Calculus*, by Augustus De Morgan, 254; *The Use of the Slide Rule*, by F. A. Halsey, 291; *Grammar School Algebra*, by William J. Milne, 291; *Standard School Algebra*, by George E. Atwood, 291; *La Mathematique Philosophie*, by C. A. Laisant, 291; *A Short Table of Integrals*, by Benjamin Osgood Peirce, 292; *Observational Geometry*, by William T. Campbell, 292; *The Story of the Philippines*, by Murat Halsted, 292; *On Several Classes of Simple Groups*, by G. A. Miller, 312; *The Construction of County Roads and Bridges*, by J. C. Nagle, 312; *Plane Trigonometry for Colleges and Secondary Schools*, by Daniel A. Murry, 312; *The Laws of Gases*, Memoirs by Robert Boyle, and E. H. Amagat, 312; *The Second Law of Thermodynamics*, Memoirs by Carnot, Clausius, and Thomson, 312; *The Fundamental Laws of Electrolytic Conduction*, Memoirs by Farady, Hittorf, and F. Kohlrausch, 312.

DIOPHANTINE ANALYSIS (See Solutions and Problems).

EDITORIALS, 22–23, 51, 94, 128, 158–159, 215–216, 252, 290, 310.

ERRATA, 24, 96, 188.